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ON SMALL GRAPHS CRITICAL WITH RESPECT TO EDGE COLOURINGS

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A simple graph is said to be of class 1 or of class 2 according as its chromatic index equals the maximum degree or is one greater. A graph of class 2 is called critical if all its proper subgraphs have smaller chromatic index. It has been conjectured by Beineke and Wilson and by Jakobsen that all critical graphs have odd order. In this paper we verify the truth of this conjecture for all graphs of order less than 12 and all graphs of order 12 and maximum degree 3. We also determine all critical graphs through order 7.

0. Introduction

The *chromatic index* $\chi_e(G)$ of a graph G is the minimum number of colours needed to colour the edges of G in such a way that no adjacent edges have the same colour. Vizing [8] proved that if G is a simple graph with maximum degree r, then $\chi_e(G) = r$ or $r+1$. We say that G is *of class* 1 if $\chi_e(G) = r$ and *of class* 2 otherwise. Furthermore, G is called *critical* or *r-critical* if it is of class 2 and each of its proper subgraphs has smaller chromatic index.

It has been conjectured by Beineke and Wilson [1], and by Jakobsen [6] that there are no critical graphs of even order. Jakobsen in fact showed that there are no 3-critical graphs of even order less than 12. One part of this paper extends his results.

In Section 1, we present some background results as well as develop some new ones relevant to our study. In Section 2, we consider critical graphs of small odd order, determining all with up to 7 vertices. Jakobsen [6] has found all 3-critical graphs with 9 vertices. In Section 3, we prove that there is no critical graph of even order less than 12 and no 3-critical graph of order 12.

In this paper, we shall, for a given graph G, let n and m denote the numbers of vertices and edges respectively, and let r and s denote the maximum and minimum

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degrees, respectively. We let f_i denote the number of vertices of degree i, and if $a_1, a_2, \ldots, a_k$ are the distinct degrees in ascending order in G, we call the *degree list* of G the expression $a_1^{n_1} a_2^{n_2} \ldots a_k^{n_k}$, where $n_k = f_{a_k}$. We denote by $d(v)$ the degree of the vertex v and we call the *total deficiency* $D(G)$ the sum $\sum_{v \in V(G)} r(G) - d(v)$. The end of a proof will be denoted by //.

1. Preliminary results

In order to facilitate the reader's referring to results when they are used later, we first state a number of theorems.

Theorem 1. *All bipartite graphs and all complete graphs of even order are of class* 1.

Theorem 2. *If G has odd order and total deficiency less than the maximum degree, then G is of class* 2.

Theorem 3. *Let G be a critical graph of order n, maximum degree r and minimum degree s. Then the total deficiency of G is at least*

$$\begin{cases} r-2 & \text{if } n \text{ is odd,} \\ 2(r-s+1) & \text{if } n \text{ is even.} \end{cases}$$

Theorem 4. *Let G be an r-critical graph.*

(a) *If v and w are adjacent vertices and $d(v) = k$, then w is adjacent to at least $r-k+1$ other vertices of degree r.*

(b) *If v and w are adjacent vertices, then $d(v) + d(w) \geq r + 2$.*

(c) *Every vertex is adjacent to at least two vertices of degree r.*

Theorem 5. *Let G be a critical graph with maximum degree r, minimum degree s and f_j vertices of degree j. Then*

(a) $f_r \geq r - s + 2$.

(b) $f_r \geq 2\sum_{j=2}^{r-1} f_j/(j-1)$.

Theorem 6. *Let G be an r-critical graph.*

(a) *If J is a set of independent edges, then $\chi_e(G \setminus J) = r$.*

(b) *If J is a 1-factor, then $G \setminus J$ is of class 2.*

(c) *G is connected and has no cut-vertex.*

(d) *If $r \geq 3$, then G is not regular.*

Theorem 7. *Let G be a 3-critical graph.*

(a) *G cannot have exactly two vertices of degree 2.*

(b) *If G is of minimum even order, then no vertex of degree 2 is on a triangle.*

Theorem 8. *Let G be an r-critical graph of even order n. If $n \leq 10$ or if $r = 3$ and $n \leq 12$, then G has a 1-factor.*

Theorem 1 is a well-known result on edge-colourings; for a proof see Wilson [10, p. 94], for example. Theorem 2 was proved by Beineke and Wilson [1] while Theorem 4 is due to Vizing [9]. A proof of Theorem 5 is found in Fiorini and Wilson [4]. As for Theorem 6, (a) and (b) are simple observations (see Fiorini and Wilson [4]), (c) was proved by Jakobsen [5] and (d) follows from Theorem 3. Similarly, Theorem 7 (a) follows from Theorem 3, while 7 (b) was proved by Jakobsen [5]. This leaves Theorems 3 and 8 to be proved.

Proof of Theorem 3. For the odd case, the result follows from Theorem 2, so we assume that G has even order. Let v be a vertex of G of minimum degree s and let v_1 be a vertex of degree r which is adjacent to v. The graph obtained by deleting the edge (v, v_1) is r-colourable, and in such a colouring, one colour is missing from v_1 and is used for some edge (v, v_2). Thus, the graph obtained from G by deleting v and adding an edge (v_1, v_2) (which may now possibly be a double edge) is r-colourable. In the case that (v_1, v_2) is a double edge with colours 1 and 2 say, we can transform the multigraph into a simple graph by effecting the following transformation:

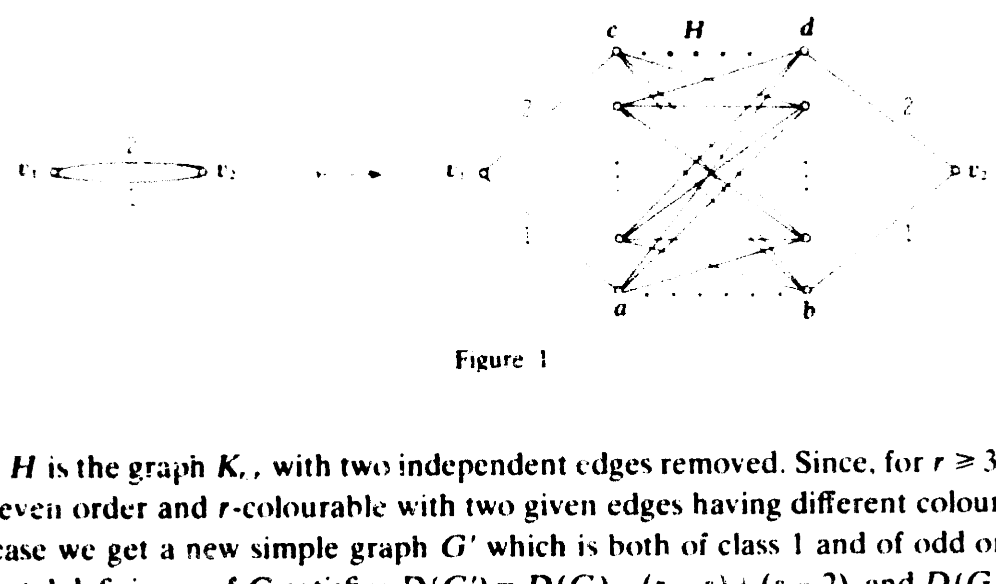


Figure 1

Here H is the graph $K_{r,r}$ with two independent edges removed. Since, for $r \geq 3$, $K_{r,r}$ is of even order and r-colourable with two given edges having different colours, in any case we get a new simple graph G' which is both of class 1 and of odd order. The total deficiency of G satisfies $D(G') = D(G) - (r - s) + (s - 2)$, and $D(G') \geq r$ by Theorem 2. Therefore, $D(G) \geq 2(r - s + 1)$ as required. //

Proof of Theorem 8. The portion of the theorem with $r = 3$ was proved by Fiorini [3]. Therefore we assume that G is an r-critical graph ($r \geq 4$) of even order $n \leq 10$ which has no 1-factor. By a theorem of Tutte [7], there is a cut-set S of k vertices for which $G \backslash S$ has more than k components of odd order. A parity argument shows that there must be at least $k + 2$ such components. We consider the various possible values of k.

$k = 2$: By Theorem 4 (a), no vertex of G can be adjacent to more than one of degree 2 so that at most one component of $G \backslash S$ has order 1. Since there must be at least four components, this is impossible.

$k = 3$: Similarly, a vertex of G can be adjacent to at most two of degree 3 (or less), so that $G \backslash S$ has at most two trivial components, which is impossible.

$k = 4$: Since $G \backslash S$ has at least six components, in this case all must be trivial. If $r = 4$, there can be at most 16 edges joining S to $G \backslash S$. Hence, of the six vertices in $G \backslash S$, at least three have degree less than 4 in G and this is impossible (as before) Therefore $r \geq 5$, so that all vertices of maximum degree are in S. If s denotes the minimum degree in G, this and Theorem 5(a) imply that $s \geq r - 2$. But since each vertex in $G \backslash S$ is adjacent only to vertices in S and every vertex in S is adjacent to at least two others in S (by Theorem 4(c)), $4r \geq 6s + 8$. As this implies $r \leq 2$, we again have a contradiction. //

It should be noted that this result can be extended to 3-critical graphs of even order not exceeding 26.

2. Odd order

It is easy to see that there are no critical graphs of order 2 or 4. The only one of order 3 is the 3-circuit, and there are only three of order 5 (shown in fig. 2). We note that each is r-critical for a different value of r. These graphs are the only ones with the corresponding degree lists, so we shall often refer to them by their lists: $2^5, 23^4, 3^2 4^3$.

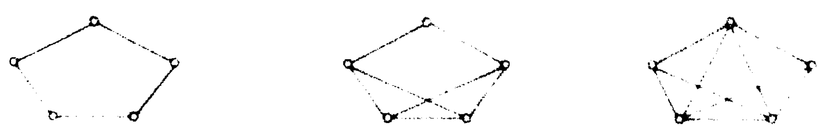


Figure 2

In our next theorem, we shall determine the degree lists of all critical graphs of order 7. The proof involves a large number of cases by looking at possibilities for maximum and minimum degrees. In many cases, the problem is reduced to the graph being in class 2 because of its total deficiency and then being r-critical b[illegible] it could contain no other r-critical graph. For simplicity, we shall refer to thi[illegible] the "critical list argument".

Theorem 9. *Let G be a connected graph of order 7. Then G is r-critical if and only if it has exactly $3r + 1$ edges.*

Proof. Let G be r-critical with minimum degree s and total deficiency t. We note that G having $3r + 1$ edges is equivalent to t equalling $r - 2$. We consider all cases (r, s) with $2 \leq s < r \leq 6$ together with the first trivial case.

Case (2.2). The 7-circuit is the only 2-critical graph and the only connected graph with $r = 2$ and $t = 0$.

Case (3.2). By Theorem 5, $f_3 \geq 2f_2$, so that the only possible degree list is 23^6. (Here and in what follows we freely use the fact that the number of vertices of odd degree must be even.) By the critical list argument, any such graph must be critical.

Case (4.2). Since in this case $f_4 \geq f_3 + 2f_2$, there are just three possible lists: 2^24^5, 23^24^4 and 24^6. Any graph with list 2^24^5 can be obtained by taking the only graph of order 6 which is regular of degree 4 and splitting one vertex into two of degree 2. Since the original graph has chromatic index 4, so has the result. Therefore 2^24^5 cannot belong to a critical graph. Because of the required adjacencies of all vertices to vertices of degree 4, there is only one possible graph with list 23^24^4 and this is 4-colourable (see fig. 3). It follows that every graph with list 24^6 is critical.

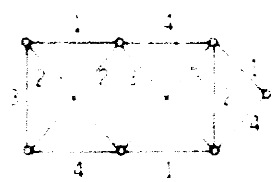


Figure 3

Case (4.3). Since we must have $f_3 \leq f_4$, there is only one possible list, 3^24^5, and any corresponding graph must be critical by the critical list argument.

Case (5.2). Since $f_5 \geq 5$ by Theorem 5, there are two lists to consider: 235^5 and 25^6. In a critical graph with the first list, the vertices of degree 2 and 3 cannot be adjacent, so that since K_6 is 5-colourable, so is this graph. Therefore the critical list argument implies that any graph with list 25^6 is critical.

Case (5.3). In this case $f_5 \geq 4$, so that 3^245^4 and 345^5 are the only possible lists. If the first belongs to G, then each vertex of degree 5 is adjacent to all three others, and since the vertices of degree 3 cannot be adjacent, the vertex of degree 4 must be adjacent to one of them. There is only one graph meeting these restrictions and it is 5-colourable (see fig. 4). As in earlier cases, we deduce that all graphs with list 345^5 are 5-critical.

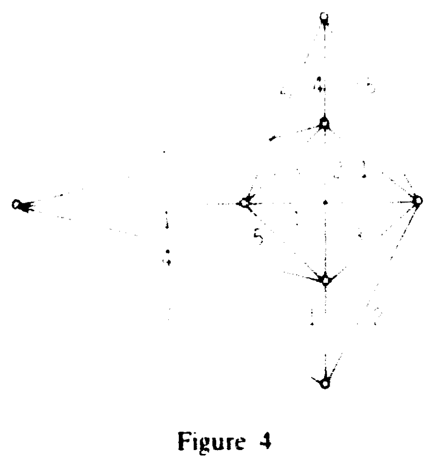


Figure 4

Case (5,4). Since $f_5 \geqslant 3$ and since the deficiency must be at least 3, there is only one list, $4^3 5^4$, and any corresponding graph is critical.

Cases (6,2) *and* (6,3). Since $f_6 \geqslant 6$ in the first case and $f_6 \geqslant 5$ in the second, there is no critical graph possible.

Case (6,4). Since $f_6 \geqslant 4$ and the deficiency is at least 4, there are three possible lists: $4^3 6^4, 4^2 6^5$ and $45^2 6^4$. The second belongs to no graph and the first only to the complement of K_3 in K_7, which is readily seen to be 6-colourable. Therefore the critical graphs here are precisely those with list $45^2 6^4$.

Case (6,5). In this case $f_6 \geqslant 3$ and, by Theorem 3, $f_5 \geqslant 4$. Therefore the only possible list is $5^4 6^3$ and all corresponding graphs are critical.

It is an easy matter to check that in each case, the r-critical graphs belong to lists with degree sum $6r + 2$. //

Corollary 1. *A connected graph of order 7 is critical if and only if its degree list is in the set* $\{2^7, 23^6, 24^6, 3^2 4^5, 25^6, 345^5, 4^3 5^4, 45^2 6^4, 5^4 6^3\}$. //

We observe that a result like the theorem also holds for $n = 3$ and $n = 5$. However, there is no comparable result for $n = 9$; there are 3-critical graphs with lists $2^3 3^6$ and 23^8. As proved by Jakobsen [6], who gives the complete collection, there are seventeen with the latter list and only one with the former. That graph is shown in fig. 5.

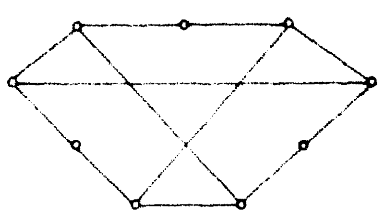


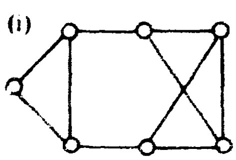
Figure 5

It is not difficult to find all critical graphs of order 7 using the corollary. We present them or their complements by list in table 1.

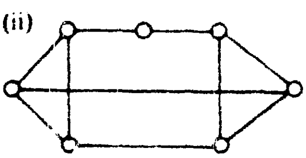
Table 1

2^7: The 7-circuit.

23^6: (i)



(ii)



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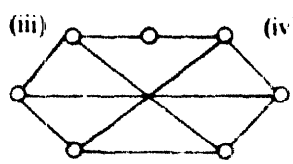

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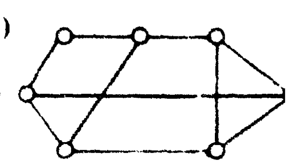
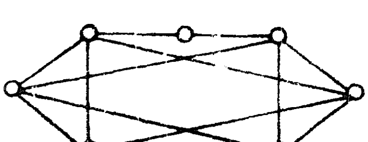
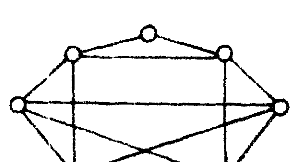


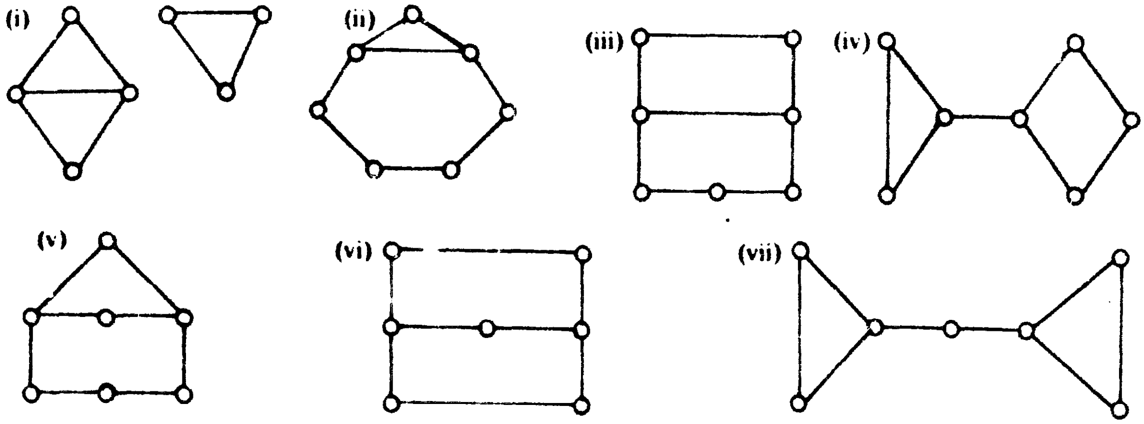
24^6: (i)



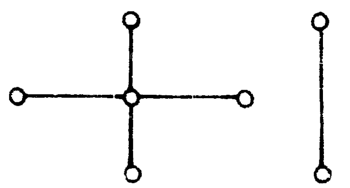
(ii)



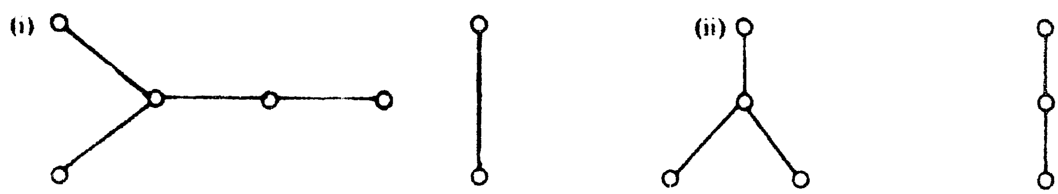
3^24^5: The complements in K_7 of the following:



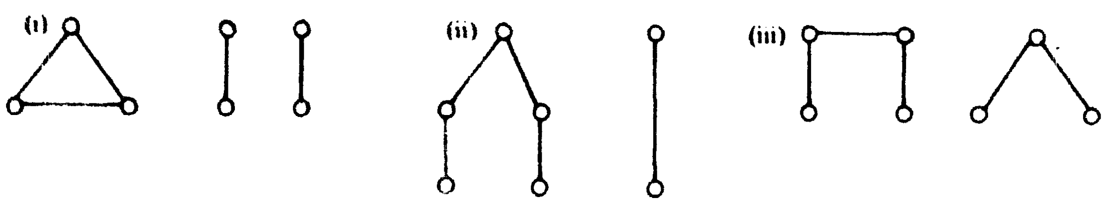
25^6: The complement in K_7 of:



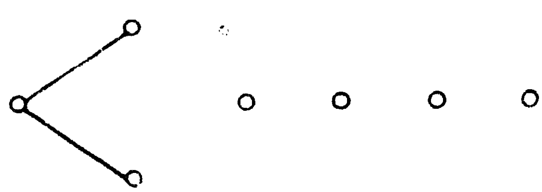
345^5: The complements in K_7 of:



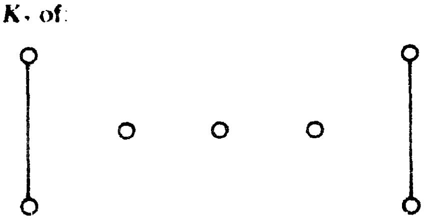
4^35^4: The complements in K_7 of:



45^26^4: The complement in K_7 of:



5^46^3: The complement in K_7 of:



3. Even order

In this section we shall prove that the smallest r-critical graph of even order must have at least 12 vertices and if $r = 3$ at least 14.

Theorem 10. *There are no critical graphs of order* 6.

Proof. Assume G is r-critical and of order 6. We consider the three possible values of r separately.

Case $r = 3$: In this case $f_2 \geq 4$ by Theorem 7 and $f_3 \geq 3$ by Theorem 5, which is impossible.

Case $r = 4$: From Theorem 5 it follows that the only possible lists are $2^2 4^4$, 24^5, $23^2 4^3$ and $3^2 4^4$. The first three contradict Theorem 3, so G must have list $3^2 4^4$. By Theorem 8, G has a 1-factor F, and the removal of F leaves a graph G' of class 2. Therefore G' must contain the 3-critical graph 23^4 of fig. 2, which is impossible for a graph with list $2^2 3^4$.

Case $r = 5$: In this case G is a subgraph of K_6 which is 5-colourable. Therefore G cannot be critical. //

Theorem 11. *There are no critical graphs of order* 8.

Proof. Assume that G is r-critical, has order 8 and minimum degree s. Since K_8 is 7-colourable and G cannot be regular, we assume $2 \leq s < r \leq 6$. By Theorem 8, G has a 1-factor F, and $G \setminus F$ contains an $(r-1)$-critical graph G'. We now consider all possible cases (r, s), relying heavily on Theorems 3 and 5 which give inequalities for the number f_j of vertices of degree j.

Case (3,2): There are no such graphs since we must have $f_3 \geq 2f_2$ and $f_2 \geq 4$, which are irreconcilable.

Case (4,2): Similarly, we must have $f_4 \geq 2f_2 + f_3$ and $2f_2 + f_3 \geq 6$, again an impossible situation.

Case (4,3): A corresponding argument implies that G must have list $3^4 4^4$. Then $G \setminus F$ must have list $2^4 3^4$ and hence G' cannot have 7 vertices. Therefore G' must have list 23^4 and any extension to G results in three vertices of degree 3 being mutually adjacent, which is impossible.

Case (5,2): Here $f_5 \geq 2f_2 + f_3$ and $3f_2 + 2f_3 + f_4 \geq 8$, so that the only possible list is $2^2 35^5$. By Theorem 4(a), such a list cannot belong to a critical graph since four of the vertices of degree 5 must be adjacent to a vertex of degree 2 and the others to vertices of degree 5.

Case (5,3): From the facts that $f_5 \geq 4$ and $2f_3 + f_4 \geq 6$, we see that $3^3 5^5$, $3^4 5^4$ and $3^2 4^2 5^4$ are the only possible lists. Since $G \setminus F$ then has list $2^3 4^5$, $2^4 4^4$ or $2^2 3^2 4^4$, we see that the order of G' cannot be 7 and hence must be 5. However, this means that G' has list $3^2 4^3$. Any extension to G requires two vertices of degree 3 to be adjacent, which is impossible.

Case (5,4): Since $f_5 \geqslant 4$, G must have list 4^45^4. Furthermore, G' must have list 3^24^4 and thus cannot be a subgraph of $G\backslash F$ (which has list 3^44^4).

Case (6,2): The deficiency of G must be at least 10 and yet f_6 must be at least 6. Thus this case cannot occur.

Cases (6,3), (6,4) *and* (6,5): It not difficult to show that the only possible lists are: $3^346^5, 4^46^4, 4^36^5, 4^35^26^4, 5^46^4$. In no case can be the list for $G\backslash F$ admit a 5-critical subgraph (which would have to be of order 7). //

Theorem 12. *There are no critical graphs of order* 10.

Proof. Assume this is not the case, and let G be an r-critical graph of order 10 with r minimum. By Theorem 8, G has a 1-factor F, and the graph $G' := G\backslash F$ has an $(r-1)$-critical subgraph H which we take to have maximum possible order. Lemmas 1 and 2 below show that this order must be 9.

Let u be the vertex of G not in H, let k be the degree of u and let s be the minimum degree in G. Then the total deficiency of $G\backslash\{u\}$ is

$$\begin{aligned} D(G\backslash\{u\}) &= D(G) - r + 2k, \\ &\geqslant 2(r-s+1) - r + 2k \quad \text{(by Theorem 3)}, \\ &\geqslant r + 2. \end{aligned}$$

Therefore $D(G'\backslash\{u\}) \geqslant r+1$. There must be at least one vertex w of maximum degree adjacent to u in G'. It follows that if t_H is the next-largest degree to $r-1$ in H, then the degree of w is between t_H and $r-2$, so that

$$D(H) \geqslant D(G'\backslash\{u\}) + 2(r-2-t_H) \geqslant 3r_H - 2t_H,$$

where $r_H (= r-1)$ is the maximum degree in H. It follows from Lemma 3 below that H must have deficiency less than this. Therefore, no such graph H can exist. //

We now prove the three lemmas used in the proof of Theorem 12. To this end, assume that G is an r-critical graph of order 10, with r minimum. If $r=3$, then $f_3 \geqslant 4$ and $f_3 \geqslant 2f_2$, so $r > 3$. By Theorem 8, G has a 1-factor F, the deletion of which leaves a graph G' which must contain an $(r-1)$-critical subgraph. Assume that H is one of maximum order. Since r is minimal, H must have odd order.

Lemma 1. *The order of H is not* 5.

Proof. Assume that the order of H is 5. Then the maximum degree r_H is either 3 or 4. Let J be the subgraph of G' induced by the five vertices not in H.

Case 1: $r_H = 3$. Then H has degree list 23^4. Since in G' each vertex must be adjacent to at least one vertex of degree 3, J must contain at least two vertices of degree 3. Moreover, no vertex of degree 3 can be adjacent to more than one vertex of degree 1. So, there are at most two vertices of degree 1. By Theorem 4 (a), we also have that if G' has a vertex of degree 1, then it has at least seven vertices of degree 3. All this implies that the only possible lists for G' are: $1^2 3^8$, $12^2 3^7$, $2^2 3^8$ and $2^4 3^6$. The first three cases clearly violate the deficiency condition of Theorem 3. The last case gives rise to the following unique graph G':

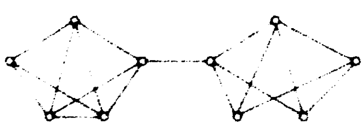
Figure 6

Moreover, since every vertex in G is to be adjacent to at least two vertices of degree 4, (u, v) must be an edge in F. But then one can clearly obtain a new 1-factor F' from G such that $G\backslash F'$ is not isomorphic to this configuration, which it must be since H is of maximum order.

Case 2: $r_H = 4$. Then H is the graph with list $3^2 4^3$. We consider sub-cases according to the number q of edges between H and J in G'.

$q = 0$: J has at least two vertices u_1, u_2 of degree 4 and these must be adjacent. Moreover, by Theorems 4 (a) and 3, the other vertices must have degrees 3, 3 and 4. So G' must consist of two disjoint copies of H. But then one can obtain a new 1-factor from G whose removal does not yield the graph $3^2 4^3$.

$q = 1$: Again, J must contain two vertices of degree 4 in G'. Now either some vertex of J with degree 4 is adjacent to some vertex in H or not. In either case, all vertices of G' have degree at least 3 and at least four have degree 3. The only possible graphs are these:



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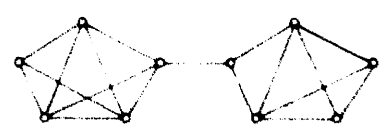

Figure 7

However, as before, we can obtain in each case a new 1-factor from G which does not contain H.

$q = 2$: If J has only one vertex of degree 4 in G', then Theorem 4 (a) and the fact that the number of vertices of odd degree has to be even, force G' to have list $3^4 4^6$. Under these conditions, G' can only be one of the following graphs:

Figure 8

Again, we can obtain a new 1-factor from G whose removal does not yield the graph $3^2 4^3$.

If J has exactly two vertices of degree 4 in G', then these must be adjacent to the vertices in H, since otherwise the deficiency condition is violated. Again by Theorem 4 (a), $s(G')$ is at least 2 and the only possible list for G' is 2^34^7. Since a vertex of degree 5 in G which is adjacent to a vertex of degree 3 has to be adjacent also to at least three other vertices of degree 5, G' can only be the following graph:

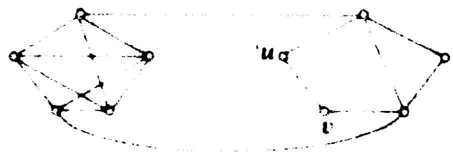

Figure 9

However, this graph violates Theorem 4 (b) since in G, $d(u) + d(v) = 6 < r(G) + 2$.

Since J cannot have three or more vertices of degree 4, by Theorem 3, the proof of the lemma is complete. //

Lemma 2. *The order of H is not 7.*

Proof. We consider individual cases of r, making repeated use of the fact that each of the three vertices u_1, u_2, u_3 in $G'\backslash H$ must be adjacent to at least one of degree $(r-1)$ in G' which does not have degree $(r-1)$ in H.

$r = 4$: H must have list 23^6 and then $f_3(G')$ is at least 8, which contradicts the deficiency condition.

$r = 5$: H must have one of the following lists: 24^6, 3^24^5. Both of these cases contradict the deficiency condition.

$r = 6$: H must have one of the following lists: 25^6, 345^5, 4^35^4. Hence, the list of G must be $x_1x_2x_36^7$ and each of u_1, u_2, u_3 has degree at most 4. Theorem 4 (a) implies that no two of these can be adjacent and so G must have list 2^36^7. Now, since no two of the u_i's can be adjacent to a common vertex, we can identify u_1, u_2 and u_3 to obtain a graph G'' which is of order 8 and regular of degree 6. G'', and hence G, is in class 1, since any such graph is obtainable from K_8 by the removal of a 1-factor.

$r = 7$: H must have one of the following lists: 45^26^4, 5^46^3. Also, in G, f_7 must be at least 6 and $f_6 + f_7$ must be at least 7 by the above conditions. By Theorem 4 (b), no two of u_1, u_2, u_3 can be adjacent. Thus, the only possible lists for G are 2^367^6 and 2^237^7. The second case violates the deficiency condition. The first yields a graph G'' obtained by identification of u_1, u_2 and u_3. G'' is 7-colourable since it is a subgraph of K_8.

This completes the proof of our second lemma. //

Lemma 3. *If G is an r-critical graph ($3 \leq r \leq 7$) of order 9 and total deficiency D, then $D < 3r - 2t$, where t is the largest degree of G less than r.*

Proof. We assume on the contrary that $D \geq 3r - 2t$ and proceed to get contradictions for all cases (r, s), $2 \leq s < r \leq 7$, where s is as usual the minimum degree of G. We note that D and r have the same parity.

Cases (5,4), (6,5) *and* (7,6): Here $D \geq r+2$ and also, since $f_r \geq 3$, $D \leq 6$, which is a contradiction.

Case (3,2): We have $f_2 \leq 3$, $f_2 = D$ and $D \geq 5$, which is impossible.

Case (4,2): Here $f_4 \geq 2f_2 + f_3 = D \geq 12 - 2t$. If $t = 3$, this implies $f_4 \geq 9$ and in turn $D \leq 4$, while if $t = 2$, $f_4 \geq 8$ and $D \leq 2$, both of which are impossible.

Case (4,3): A contradiction follows from the inequalities $f_3 \leq f_4$ (so $f_3 \leq 4$) and $f_3 = D \geq 6$.

Case (5,2): On the one hand, $D \geq 15 - 2t$, and on the other hand, $D \leq 14 - t$ (since $f_5 \geq 5$). It follows that if $t = 2$, $D = 11$; if $t = 3$, $D = 9$ or 11; and if $t = 4$, $D = 7$ or 9. The only lists which meet these conditions are 23^35^5, 234^25^5 and 2^245^6. It is not difficult to show that none of these can meet the adjacency conditions of Theorem 4 for a 5-critical graph.

Case (5,3): Here $t = 4$, so $D \geq 7$. Also $f_5 \geq 4$, so that $D \leq 9$. The only possible lists are therefore $3^24^35^4$, 3^445^4 and 3^345^5, none of which is constructible as a 5-critical graph.

Case (6,2): Since $f_6 \geq 6$, we have $D \leq 14 - t$ and $D \geq 18 - 2t$, so t is 4 or 5. If $t = 4$, $D = 10$ and the only list is 2^246^6, which cannot belong to a 6-critical graph. If $t = 5$, $D = 8$ and the only list is 2356^6. Such a graph has a hamiltonian circuit by Pósa's Theorem (see, for example [2, p. 211]) so there is a set F of four independent edges not meeting the vertex of degree 2. Then $G \setminus F$ has list 2^245^6 and must contain a 5-critical subgraph T. The order of T cannot be 9, so T must have 25^6, 345^5 or 4^35^4 as its list. However, each of these requires the two vertices of degree 2 to be adjacent in $G \setminus F$, which is impossible.

Case (6,3): Arguments similar to those given above show that the only possible lists are 3^46^5, 3^356^5, $3^24^26^5$, $3^25^26^5$ and 34^256^5. In each of the first four cases, there must be at least twelve edges from vertices of degree 6 to other vertices, and yet each vertex of degree 6 must be adjacent to four others. This is clearly impossible. The graphs with list 34^256^5 are handled using four independent edges as in the preceding case.

Case (6,4): In this case, $f_6 \geq 4$, $D = 2f_4 + f_5$ and $D \geq 18 - 2t$. The only possibilities are 4^56^4 and $4^35^26^4$, but since any vertex adjacent to one of degree 4 must also be adjacent to at least three of degree 6, it is clear that such a critical graph cannot exist.

Case (7,2): Since $f_7 \geq 7$, $D \leq 12 - t$. But we must also have $D \geq 21 - 2t$, which is impossible.

Case (7,3): Here $f_7 \geq 6$, so $D \leq 15 - t$. Again $D \geq 21 - 2t$, so that $t = 6$ and 3^267^6 is the only possible list. It is readily seen that the vertices of degree 3 are not adjacent and that their identification results in a simple subgraph of K_8, which is thus 7-colourable.

Cases (7,4) *and* (7,5): Because of bounds on f_7, we have $D \leq 16 - t$ in the first case and $D \leq 15 - t$ in the second. These inequalities, together with $D \geq 21 - 2t$, restrict D and t so that the only possible lists are $4^3 5 7^5$, $4^2 5 6 7^5$ and $5^4 6 7^4$. Any such graph can be shown to have four independent edges whose removal leaves a graph which cannot have a 6-critical subgraph. (cf. Case (6.2))

This completes the proof of the lemma. //

Finally we prove a result which extends the earlier work of Jakobsen [6].

Theorem 13. *There are no 3-critical graphs of order* 12.

Proof. Assume on the contrary, that G is a 3-critical graph of order 12. It follows from Theorems 4 and 7 that G has exactly four vertices of degree 2 and eight of degree 3. Furthermore, the vertices of degree 2 must be at distance at least 3 apart. By Theorem 8, G has a 1-factor F, whose removal leaves a class 2 graph G' with list $1^4 2^8$. It follows that since G' must contain an odd circuit it is the graph in fig. 10, in which the pairs joined by dotted lines cannot be adjacent in G, by Theorem 7 (b).

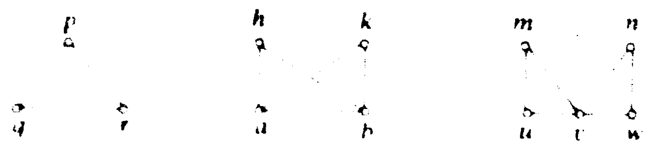

Figure 10

Since every vertex of degree 3 must be adjacent to one of degree 2, in G, a, b, u, v, w must generate a 5-circuit. Consequently, p, q or r is adjacent to m or n, say p to m, without loss of generality. But then the edges (p, m), (q, r), (h, a), (k, b), (n, w) and (u, v) form a 1-factor whose removal leaves a graph with no odd circuits. //

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